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Direct Gauging of the Poincaré Group. II

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This paper continues the study of direct gauge theory of the Poincaré group P_{10}. The meanings and implications of transformations induced by the local action of P_{10} are studied, and transformation rules for all field quantities are derived for the local action of P_{10} in a sufficiently small neighborhood of the identity. These results lead directly to a system of fundamental partial differential equations that are both necessary and sufficient for invariance of the "free field" Lagrangian density. Homogeneity arguments and the classical theory of invariants are used to obtain the most general "free field" Lagrangian density. Gauge conditions are shown to imply coordinate conditions, and an algebraic system of antiexact gauge conditions is implemented. The underlying Minkowski space, M_4, and the resulting Riemann-Cartan space, U_4, become attached at their "centers," as do their respective frame and coframe bundles. Weak constraints of vanishing torsion are studied. All field quantities are shown to be determined in terms of the compensating 1-forms for the Lorentz sector alone provided an explicit system of integrability conditions is satisfied. Field equations of the Einstein type are shown to result.

1. INTRODUCTION

A direct gauge theory of the Poincaré group, P_{10}, was presented in a previous communication (Edelen, 1985a). This paper will be referred to as I and equations will be cited from I by hyphenation with I (equation 37 of I will be written I-37).

The basic idea in I was the realization of P_{10} as a matrix Lie group of automorphisms of an affine set in a five-dimensional vector space so that the Yang-Mills constructs of minimal replacement and minimal coupling could be used without modification. This fairly simple-minded procedure leads to a system of compensating fields of 1-forms W^α for the local action of the Lorentz sector $L(4, R)$ and ϕ^i for the semidirect product action of the translation group $T(4)$. The field equations for the $T(4)$ compensating

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fields were driven by current 3-forms that obtained from momentum-energy 3-forms of the matter fields under minimal replacement and by self-sources determined by the free field Lagrangian density. Further, the field equations for the compensating fields of the $L(4, R)$ sector could be decoupled in the sense that the gauge momentum energy currents did not contribute an orbital part to the spin currents.

Noting that minimal replacement applied to the basis 1-forms dx^i on the underlying Minkowski space, M_4, gave distortion 1-forms B^i, the differential system constructed from the B's gave $DB^i = \Sigma^i$, where the Σ's are the 2-forms of Cartan torsion of the differential system. Minimal replacement applied to the line element of M_4 gave a well-defined metric tensor and its associated differential concomitants. Minimal replacement thus led to fields on M_4 that define all relevant structure associated with a Riemann–Cartan space U_4, but with a number of essential differences from work previously reported in the literature. These differences are primarily due to the occurrence of both the $T(4)$ and the $L(4, R)$ compensating fields in the distortion 1-forms, and that the distortion 1-forms play the same roles as the coframes in previous work where they are identified with the compensating fields for the $T(4)$ sector alone.

The differences between the direct gauge theory of the Poincaré group reported in I and previously published gauge theories of gravity and spin are sufficiently prominent that further analysis seems both useful and necessary. A summary of relevant results from I is given in Section 2 so that this paper will be relatively self-contained. Section 3 studies the meaning and implications of transformations induced by the local action of P_{10}. In particular, the transformation rules for all field quantities are derived for local transformations of P_{10} in a sufficiently small neighborhood of the identity (infinitesimal transformations). These results provide the basis for obtaining a system of fundamental partial differential equations in Section 4 whose solutions determine the most general free field Lagrangian density that is invariant under local action of P_{10}. Section 5 obtains the general solution of these equations by simple homogeneity arguments and the classical theory of invariants. With these results out of the way, Section 6 shows that coordinate conditions and gauge conditions are equivalent and implements a system of antiexact gauge conditions. These gauge conditions are used in Section 7 in conjunction with the weak constraints of vanishing Cartan torsion to show that equations of the Einstein-type result.

2. DIRECT POINCARÉ GAUGE THEORY

Pertinent results from the direct gauge theory of the Poincaré group, $P_{10} = L(4, \mathrm{R}) \triangleright T(4)$, are summarized in this section.

Let M_4 be Minkowski space with the standard coordinate cover $\{x^i | 1 \leq i \leq 4\}$, volume 4-form μ, and metric tensor h_{ij}. Local action of P_{10} generates the transformations

$$'x^i = L^i_j(x^k)x^j + t^i(x^k) \tag{1}$$

where $\mathbf{L}$ is a smooth position-dependent Lorentz transformation matrix and t^i, $1 \leq i \leq 4$ are smooth functions. A system of canonical parameters for the action of P_{10} is denoted by $\{u^\alpha, u^i | 1 \leq \alpha \leq 6, 1 \leq i \leq 4\}$ and the corresponding fields of Yang–Mills compensating 1-forms are $\{W^\alpha(x^k), \phi^i(x^k)\}$. The local action of $L(4, R)$ is compensated for by the W's while the local semidirect product action of $T(4)$ is compensated by the ϕ's.

Realization of P_{10} as a subgroup of $GL(5, R)$ that maps an affine set in V_5 into itself gives rise to the gauge covariant derivative

$$D\mathbf{x} = d\mathbf{x} + \mathbf{\Gamma x} + \boldsymbol{\omega} \tag{2}$$

where

$$\mathbf{\Gamma} = W^\alpha \mathbf{l}_\alpha \tag{3}$$

is the 1-form valued connection matrix for the $L(4, R)$ component and

$$\boldsymbol{\omega} = \phi^i \mathbf{e}_i \tag{4}$$

is the 1-form valued connection matrix for the semidirect product action of $T(4)$. Here $\{\mathbf{l}_\alpha | 1 \leq \alpha \leq 6\}$ is a basis for the matrix Lie algebra of $L(4, R)$ and $\{\mathbf{e}_i | 1 \leq i \leq 4\}$ is a basis for $T(4)$. The corresponding curvature quantities are

$$\boldsymbol{\theta} = \theta^\alpha \mathbf{l}_\alpha, \qquad \mathbf{\Omega} = \Omega^i \mathbf{e}_i \tag{5}$$

where

$$\theta^\alpha = dW^\alpha + C^\alpha_{\beta\gamma} W^\beta \wedge W^\gamma/2 \tag{6}$$

$$\Omega^i = d\phi^i + C^i_{\alpha j} W^\alpha \wedge \phi^j \tag{7}$$

Let $\mathcal{M}$ denote the operation of minimal replacement. We then have

$$\mathcal{M}(dx^i) = Dx^i = B^i \tag{8}$$

where

$$B^i = B^i_j\, dx^j, \qquad B^i_j = \delta^i_j + W^\alpha_j l^i_{\alpha k} x^k + \phi^i_j \tag{9}$$

are the distortion 1-forms for the Poincaré group. They serve to define the system of fundamental coframes for the theory. This is made evident from

$$dS^2 = \mathcal{M}(h_{ij}\, dx^i\, dx^j) = g_{mn}\, dx^m\, dx^n \tag{10}$$

$$g_{mn} = B^i_m h_{ij} B^j_n \tag{11}$$

The duals of the distortion 1-forms are defined by

$$b_i \lrcorner B^j = \delta^j_i, \qquad b_i = b^k_i \partial_k \tag{12}$$

where

$$B^i_j b^j_k = b^i_j B^j_k = \delta^i_k \tag{13}$$

and

$$B = \det(B^i_j) \neq 0 \tag{14}$$

Thus $\{b_i | 1 \leq i \leq 4\}$ is a basis for $T(M_4)$ and we have

$$g^{ij} = b^i_m h^{mn} b^j_n, \qquad g^{ij} g_{jk} = \delta^i_k \tag{15}$$

The distortion 1-forms also give rise to the Cartan torsion of the differential system generated by the distortion 1-forms through the relations

$$DB^i = \Sigma^i \tag{16}$$

The explicit evaluation is

$$\Sigma^i = \theta^\alpha l^i_{\alpha j} x^j + \Omega^i. \tag{17}$$

Minimal replacement and minimal coupling give the action 4-form $\mathcal{M}(\bar{L}\mu) + V\mu = (LB + V)\mu$, where

$$\bar{L}(x^i, \Psi^A, \partial_i \Psi^A)$$

is the Lagrangian for the matter fields Ψ^A, and

$$V(x^i, W^\alpha_i, \phi^i_j, \theta^\alpha_{ij}, \Sigma^k_{ij})$$

(denoted as $\bar{V}$ in I) is a scalar density valued function of the indicated arguments that is invariant under the local action of P_{10}. Field equations are then computed in the standard way as Euler–Lagrange equations for the Lagrangian density $LB + V$.

The reader is referred to I for the field equations of the matter fields since they are not directly relevant to the considerations of this paper; they do not involve the "free field" Lagrangian density V. Variations of the action with respect to the ϕ fields and the W fields lead to the field equations for the compensating gauge fields. These equations are most easily written through use of the following notation:

$$\mu_i = \partial_i \lrcorner \mu, \qquad \mu_{ij} = \partial_i \lrcorner \mu_j = -\mu_{ji} \tag{18}$$

$$L^i_A = \partial L / \partial y^A_i, \qquad y^A_i = \mathcal{M}(\partial_i \Psi^A) \tag{19}$$

$$G^{ij}_k = -G^{ji}_k = \partial V / \partial \Sigma^k_{ij}, \qquad S^i_k = (\partial V / \partial \phi^k_i)|_{\theta,\Sigma} \tag{20}$$

$$H^{ij}_\alpha = -H^{ji}_\alpha = (\partial V / \partial \theta^\alpha_{ij})|_\Sigma, \qquad S^i_\alpha = (\partial V / \partial W^\alpha_i)|_{\theta,\Sigma} \tag{21}$$

$$T^i_j = L^i_A y^A_j - \delta^i_j L, \qquad T_j = T^i_j B b^k_i \mu_k \tag{22}$$

where T_j are the gauge momentum-energy 3-forms of the matter fields (they obtain from minimal replacement applied to the 3-forms of momentum-energy of the matter fields). Variation with respect to the ϕ's gives the field equations

$$d\{G_k^{ij}\mu_{ij}\} - W^\alpha l_{\alpha k}^m \wedge \{G_m^{ij}\mu_{ij}\} = -T_k + S_k^i\mu_i, \qquad 1 \le k \le 4 \tag{23}$$

Finally, variation with respect to the W's gives

$$\begin{aligned} d\{H_\alpha^{ij}\mu_{ij}\} - W^\gamma C_{\gamma\alpha}^\beta \wedge \{H_\beta^{ij}\mu_{ij}\} \\ = (Bb_k^i L_A^k \mu_i) M_{\alpha B}^A \Psi^B + (S_\alpha^i - l_{\alpha m}^k x^m S_k^i)\mu_i \\ - B^k l_{\alpha k}^m \wedge \{G_m^{ij}\mu_{ij}\}, \qquad 1 \le \alpha \le 6 \end{aligned} \tag{24}$$

Here, the M's are the explicit representation matrices for the induced action of the six infinitesimal generating transformations of $L(4, R)$ acting on the matter fields.

3. FUNDAMENTAL TRANSFORMATION RELATIONS

Local action of the gauge group P_{10} on M_4 gives

$$'x^i = L_j^i(x^k)x^j + t^i(x^k) \tag{25}$$

Since the four functions t^i may be chosen arbitrarily, the local action of P_{10} induces the class of all smooth transformations of the x's (i.e., the Lorentz part can be absorbed into the translation part). It would thus appear that the Lorentz part becomes lost as soon as P_{10} acts locally. This is not the case as we now proceed to show.

The distortion 1-forms have the gauge transformation law (see I-33)

$$'\mathbf{B} = \mathbf{LB} \tag{26}$$

When this is written out in component form, we have

$$'B_m^i \, d'x^m = L_j^i B_k^j \, dx^k$$

and hence use of (25) gives

$$'B_j^i = L_k^i B_m^k \frac{\partial x^m}{\partial' x^j} \tag{27}$$

Similarly, since the b's are the inverses of the B's, the b's have the explicit gauge transformation laws

$$'b_i^k = \overset{-1}{L}{}_i^j b_j^m \frac{\partial' x^k}{\partial x^m} \tag{28}$$

These formulas show that the lower index on the B's and the upper index on the b's have the tensor law of transformation under arbitrary coordinate transformations generated by the local action of $T(4)$. On the other hand, the upper index on the B's and the lower index on the b's undergo pure gauge transformations that are generated by the local Lorentz matrix $\mathbf{L}(x^k)$ and its inverse, respectively. Thus, both the local $T(4)$ and the local $L(4, R)$ parts are represented in the transformation laws for the coframes and frames induced by minimal replacement.

The results just noted have far-reaching consequences. It is natural to define the images of the induced metric tensor and its inverse by

$$`g_{km} = `B^i_k h_{ij} `B^j_m, \qquad `g^{km} = `b^k_i h^{ij} `b^m_j \tag{29}$$

in view of the group property of P_{10}. When the above transformation formulas are substituted into (29), and we use

$$L^i_m h_{ij} L^j_n = h_{mn}, \qquad \overset{-1}{L}{}^m_i h^{ij} \overset{-1}{L}{}^n_j = h^{mn} \tag{30}$$

we obtain

$$`g_{ij} = g_{mn} \frac{\partial x^m}{\partial `x^i} \frac{\partial x^n}{\partial `x^j}, \qquad `g^{ij} = g^{mn} \frac{\partial `x^i}{\partial x^m} \frac{\partial `x^j}{\partial x^n} \tag{31}$$

The metric tensor and its inverse participate in the general coordinate transformations that result from the local action of $T(4)$, but are invariant with respect to the local action of $L(4, R)$. This is just as it should be, however, if the metric tensor and its inverse are to play a role similar to that in general relativity.

There is another aspect of these relations that should be noted. The definitions of the image g's given by (29) use the components of the metric tensor on M_4 for both the original and the image g's. This may be viewed as a statement that the original geometric structure on M_4 does not participate in the general coordinate transformations generated by the local action of $T(4)$. This, in turn, says that the metric structure represented by the g's is the metric structure of a new space U_4 that arises out of M_4 by minimal replacement, but is distinct from M_4. We may thus view the space M_4 as an immutable, four dimensional, flat space that is a reference space for the structures and fields that arise through minimal replacement. Two spaces are thus necessarily involved. The distinct structures and interplays that arise through use of this viewpoint have many facets in common with bimetric theories (Rosen, 1963, 1980).

We confine attention for the remainder of this section to the properties of local Poincaré transformations in an infinitesimal neighborhood of the identity. To this end, let $\{\Delta u^\alpha, \Delta u^i | 1 \le \alpha \le 6, 1 \le i \le 4\}$ be a system of position dependent canonical parameters in an appropriately chosen infinitesimal neighborhood of $u^\alpha = 0$, $u^i = 0$. Only quantities of first order in these small quantities will be retained. Equalities that are satisfied to first order will be denoted with the symbol $\simeq$. Under these conditions, (25) reduce to

$$`x^i \simeq x^i + \Delta u^\alpha l^i_{\alpha j} x^j + \Delta u^i \tag{32}$$

Accordingly, we have

$$d`x^i \simeq dx^i + \Delta^i_k \, dx^k, \qquad \Delta^i_k = \partial_k (\Delta u^\alpha) l^i_{\alpha j} x^j + \Delta u^\alpha l^i_{\alpha k} + \partial_k (\Delta u^i) \tag{33}$$

and hence

$$`\mu \simeq (1 + \Delta^k_k)\mu \tag{34}$$

Under local action of P_{10}, the results obtained in I show that

$$`\mathbf{\Gamma} = \mathbf{L}\mathbf{\Gamma}\mathbf{L}^{-1} - d\mathbf{L}\,\mathbf{L}^{-1} \tag{35}$$

$$`\boldsymbol{\omega} = \mathbf{L}\boldsymbol{\omega} - `\mathbf{\Gamma}\mathbf{t} - d\mathbf{t} \tag{36}$$

$$`\boldsymbol{\theta} = \mathbf{L}\boldsymbol{\theta}\mathbf{L}^{-1}, \qquad `\mathbf{\Sigma} = \mathbf{L}\mathbf{\Sigma} \tag{37}$$

when (3) is used, (35) shows that we have

$$`W^\alpha \simeq W^\alpha - d\Delta u^\alpha + \Delta u^\beta C^\alpha_{\beta\gamma} W^\gamma$$

Expanding in terms of the appropriate bases for 1-forms gives us

$$`W^\alpha_i \, d`x^i \simeq (W^\alpha_j - \partial_j \Delta u^\alpha + \Delta u^\beta C^\alpha_{\beta\gamma} W^\gamma_j) \, dx^j.$$

However,

$$`W^\alpha_i \, d`x^i \simeq `W^\alpha_i (dx^i + \Delta^i_k \, dx^k)$$

by (33), and

$$`W^\alpha_i \, \Delta^i_k \simeq W^\alpha_i \, \Delta^i_k$$

Combining the various terms, it follows that

$$`W^\alpha_j \simeq W^\alpha_j + \Delta W^\alpha_j \tag{38}$$

$$\Delta W^\alpha_j = -\partial_j \, \Delta u^\alpha + \Delta u^\beta C^\alpha_{\beta\gamma} W^\gamma_j - W^\alpha_i \, \Delta^i_j \tag{39}$$

Noting that

$$L^i_j \simeq \delta^i_j + \Delta u^\alpha l^i_{\alpha j}$$

similar arguments give

$$\begin{gathered} `\phi^j_k \simeq \phi^j_k + \Delta\phi^j_k, \qquad `\theta^\gamma_{ij} \simeq \theta^\gamma_{ij} + \Delta\theta^\gamma_{ij} \\ `\Sigma^k_{ij} \simeq \Sigma^k_{ij} + \Delta\Sigma^k_{ij} \end{gathered} \tag{40}$$

where

$$\Delta\phi^j_k = (\Delta u^\alpha \phi^i_k - \Delta u^i W^\alpha_k) l^j_{\alpha i} - \partial_k \Delta u^j - \phi^j_i \Delta^i_k \tag{41}$$

$$\Delta\theta^\gamma_{ij} = \Delta u^\alpha C^\gamma_{\alpha\beta}\theta^\beta_{ij} - \theta^\gamma_{ik}\,\Delta^k_j - \theta^\gamma_{kj}\,\Delta^k_i \tag{42}$$

$$\Delta\Sigma^k_{ij} = \Delta u^\alpha \Sigma^m_{ij} l^k_{\alpha m} - \Sigma^k_{im}\,\Delta^m_j - \Sigma^k_{mj}\,\Delta^m_i \tag{43}$$

4. INVARIANCE CONDITIONS

The only theoretical restriction on the choice of the free field Lagrangian is that it lead to a free field action 4-form $V\mu$ that is invariant under the local action of P_{10}; that is,

$$V('x, 'W, '\phi, '\theta, '\Sigma)'\mu = V(x, W, \phi, \theta, \Sigma)\mu \tag{44}$$

In view of the group property, satisfaction of (44) for all elements of P_{10} in a neighborhood of the identity will be both necessary and sufficient to guarantee satisfaction of (44) for all local Poincaré transformations. When (32), (34), (38), and (40) are used, we obtain

$$V\,\Delta^k_k + (\partial_i V)\,\Delta x^i + S^i_\alpha\,\Delta W^\alpha_i - S^i_k\,\Delta\phi^k_i + H^{ij}_\alpha\,\Delta\theta^\alpha_{ij} + G^{ij}_k\,\Delta\Sigma^k_{ij} = 0 \tag{45}$$

When the various Δ quantities are evaluated by the formulas given in the previous section, a page long expression results that is linear, homogeneous in the Δu's and their first derivatives. Since this expression must be satisfied for all smooth evaluations of the Δu's and their derivatives, the coefficient of each Δu and of each derivative must vanish separately.

The coefficient of Δu^i gives

$$0 = \partial_i V - S^k_j W^\alpha_k l^j_{\alpha i} \tag{46}$$

while the Δu^α terms lead to the requirements

$$\begin{aligned} 0 = {} & (\partial_i V) l^i_{\alpha j} x^j + S^i_\gamma (W^\beta_j C^\gamma_{\alpha\beta} - W^\gamma_i l^i_{\alpha j}) + S^k_j (\phi^i_k l^j_{\alpha i} - \phi^j_i l^i_{\alpha k}) \\ & + H^{kj}_\gamma (C^\gamma_{\alpha\beta}\theta^\beta_{kj} - l^i_{\alpha k}\theta^\gamma_{ij} - l^i_{\alpha j}\theta^\gamma_{ki}) \\ & + G^{nm}_k (\Sigma^j_{nm} l^k_{\alpha j} - \Sigma^k_{im} l^i_{\alpha n} - \Sigma^k_{ni} l^i_{\alpha m}) \end{aligned} \tag{47}$$

From the terms involving $\partial_s\,\Delta u^r$, we have

$$0 = V\delta^s_r - S^s_\gamma W^\gamma_r - S^s_j(\delta^j_r + \phi^j_r) - 2H^{sk}_\gamma \theta^\gamma_{rk} - 2G^{sj}_k \Sigma^k_{rj} \tag{48}$$

while the terms involving $\partial_r\,\Delta u^\rho$ give us

$$0 = -S^r_\rho + \{V\delta^r_i - S^r_\gamma W^\gamma_i - S^r_j \phi^j_i - 2H^{rk}_\gamma \theta^\gamma_{ik} - 2G^{rj}_k \Sigma^k_{ij}\}^i_{\rho m} x^m \tag{49}$$

Equations (46)-(49) are the conditions that any free field Lagrangian for the P_{10} gauge theory must satisfy.

There is an important result that obtains immediately from (48) and (49). Elimination of the common terms between these two equations gives the relations

$$S^r_\rho = S^r_i l^i_{\rho j} x^j \tag{50}$$

The S's were shown in I to be self-sources for the ϕ fields and the W fields [see (23), (24) and the definitions (20), (21)]. Further, the decoupling of momentum–energy and spin currents given in I gave the decoupled spin equations (24) in which the self sources occur only in the combination

$$\hat{S}_\alpha = (S^i_\alpha - l^k_{\alpha m} x^m S^i_k)\mu_i \tag{51}$$

When (50) is substituted into (51), all self-source terms in the spin equations vanish identically. It thus follows that *local P_{10}-gauge invariance of the free field Lagrangian leads to identically zero self-source spin currents*!

There is another important implication of (50). If these relations are put back into (48) and we use (9), we have

$$S^s_j B^j_r + 2H^{sk}_\gamma \theta^\gamma_{rk} + 2G^{sj}_k \Sigma^k_{rj} = V\delta^s_r \tag{52}$$

Remembering that the S's, H's, and G's are derivatives of V with respect to the ϕ's, θ's, and Σ's, respectively, we see that (52) is a first-order partial differential equation for the determination of V as a function of these arguments. We note however that it is the B's rather than the ϕ's that multiply the S's in (52). Further, it follows from (9) that

$$\phi^i_j = B^i_j - \delta^i_j - W^\alpha_j l^i_{\alpha k} x^k \tag{53}$$

and hence the ϕ's can be eliminated in favor of the B's with no loss of generality. We therefore use (53) to set

$$\bar{V}(x, W, B, \theta, \Sigma) = V(x, W, \phi, \theta, \Sigma) \tag{54}$$

With this notation, we have

$$\bar{S}^m_k = \partial \bar{V}/\partial B^k_m = \partial V/\partial \phi^k_m = S^m_k \tag{55}$$

$$H^{ij}_\alpha = \bar{H}^{ij}_\alpha, \qquad G^{ij}_k = \bar{G}^{ij}_k \tag{56}$$

$$\partial_i V = \partial_i \bar{V} + \bar{S}^j_k W^\alpha_j l^k_{\alpha i} \tag{57}$$

$$S^i_\alpha = \bar{S}^i_\alpha + \bar{S}^i_k l^k_{\alpha j} x^j \tag{58}$$

When (50) and (55) are substituted into (58), we have

$$0 = \bar{S}^i_\alpha = \partial \bar{V}/\partial W^\alpha_i \tag{59}$$

Similarly, when the above results are substituted into (46), we obtain

$$0 = \partial_i \bar{V} \tag{60}$$

It thus follows from (59) and (60) that P_{10}*-gauge invariance can only obtain for*

$$\bar{V}=\bar{V}(B^i_j,\theta^\alpha_{ij},\Sigma^k_{ij}) \tag{61}$$

which is a considerable reduction in the number of possible arguments. Further, and of greater significance is the fact that the surviving arguments, namely, the B's, the θ's, and the Σ's all transform linearly and homogeneously under gauge transformations.

The reduced form of the free field Lagrangian given by (61) is such that two of the systems of invariance conditions are satisfied identically. Those not identically satisfied are

$$\bar{S}^s_j B^j_r+2\bar{H}^{sk}_\gamma\theta^\gamma_{rk}+2\bar{G}^{sj}_k\Sigma^k_{rj}=\bar{V}\delta^s_r \tag{62}$$

$$\bar{S}^k_j(W^\beta_k l^j_{\beta i}l^i_{\alpha m}x^m+\phi^i_k l^j_{\alpha i}-\phi^j_i l^i_{\alpha k})+\bar{H}^{kj}_\gamma(C^\gamma_{\alpha\beta}\theta^\beta_{kj}-l^i_{\alpha k}\theta^\gamma_{ij}-l^i_{\alpha j}\theta^\gamma_{ki})$$
$$+\bar{G}^{nm}_k(\Sigma^j_{nm}l^k_{\alpha j}-\Sigma^k_{im}l^i_{\alpha n}-\Sigma^k_{ni}l^i_{\alpha m})=0 \tag{63}$$

5. FREE FIELD LAGRANGIANS

The problem of finding all free field Lagrangian for P_{10} gauge theory is that of constructing all solutions

$$\bar{V}(B^i_j,\theta^\alpha_{ij},\Sigma^k_{ij})$$

of the system of first-order, quasilinear partial differential equations (62), (63). If we set $s=r$ and sum over the repeated index in (62), we obtain

$$\bar{S}^r_j B^j_r+2\bar{H}^{rk}_\gamma\theta^\gamma_{rk}+2\bar{G}^{rj}_k\Sigma^k_{rj}=4\bar{V} \tag{64}$$

where $\{\bar{S},\bar{H},\bar{G}\}$ are the derivatives of $\bar{V}$ with respect to $\{B,\theta,\Sigma\}$, respectively. This equation is of the form

$$(x\partial_x+2y\partial_y+2z\partial_z)f=4f$$

which has the general solution

$$f=x^4\Psi(yx^{-2},zx^{-2})$$

Thus, since the correspondence is $\{x,y,z\}$: $\{B,\theta,\Sigma\}$, we need to construct quantities that are homogeneous of degree 4 in the B's, quantities that are jointly homogeneous of degree -2 in the B's and of degree 1 in the θ's, and quantities that are jointly homogeneous of degree -2 in the B's and of degree 1 in the Σ's.

The only scalar density valued function of the B's that is homogeneous of degree 4 in the B's is

$$B=\det(B^i_j)=[-\det(g_{ij})]^{1/2} \tag{65}$$

Thus since any scalar density is the product of a scalar with any one specific scalar density, we may set

$$\bar{V} = B\Pi(B^i_j,\ \theta^\alpha_{ij}, \Sigma^k_{ij}) \tag{66}$$

where Π is a P_{10}-invariant scalar-valued function of its indicated arguments.

The search for Π's is significantly simplified by noting that the previous discussion requires Π to depend on its arguments only through functions that are either jointly homogeneous of degree $-2r$ in the B's and homogeneous of degree r in the θ, s, or jointly homogeneous of degree $-2r$ in the B's and homogeneous of degree r in the Σ's. Now, b^i_j are homogeneous of degree -1 in the B's since they are the components of the inverse of B^i_j. Further, the b's serve to define the natural frame vectors

$$b_i = b^j_i \partial_j \tag{67}$$

for the P_{10} gauge theory. They may therefore be used to construct scalars from the 2-forms θ^α and Σ^k by inner multiplication.

With these ideas in mind, we construct the quantities

$$U^{i\alpha}_j = h^{im} b_m \lrcorner b_j \lrcorner \theta^\alpha = h^{im} b^u_m b^v_j \theta^\alpha_{uv} \tag{68}$$

which are jointly homogeneous of degree -2 in the B's and homogeneous of degree 1 in the θ's. If we set

$$A^i_j = U^{i\alpha}_k l^k_{\alpha j} \tag{69}$$

and use (28), (30), and (37), a direct calculation shows that

$$'A^m_n = L^m_i A^i_j \overset{-1}{L}{}^j_n \qquad ('\mathbf{A} = \mathbf{L}\mathbf{A}\mathbf{L}^{-1}) \tag{70}$$

under P_{10} gauge transformations. Thus, the A's transform as scalars under the general coordinate transformations generated by local action of $T(4)$, while the local action of $L(4, R)$ obtains through the adjoint action of the associated Lorentz matrix of functions, $\mathbf{L}$. It is thus a simple matter to construct the following list of P_{10} invariant scalar functions:

$$\alpha_1 = \mathrm{tr}(\mathbf{A}), \qquad \alpha_2 = \mathrm{tr}(\mathbf{A}^2), \qquad \alpha = \mathrm{tr}(\mathbf{A}^3), \qquad \alpha_4 = \mathrm{tr}(\mathbf{A}^4) \tag{71}$$

Any other P_{10} invariant scalar-valued function of $\mathbf{A}$ can be expressed as a function of the four invariants (71) by the Cayley-Hamilton theorem.

Although we have the homogeneous quadratic invariant α_2, there are others that can be constructed from the U's that are not expressible in terms of α_2 and the square of α_1. In order to obtain these additional invariants, we construct the quantities

$$E^i_j = U^{p\alpha}_q U^{q\beta}_p l^i_{\alpha m} l^m_{\beta j} = g^{ap} g^{bq} \theta^\alpha_{aq} \theta^\beta_{bp} l^i_{\alpha m} l^m_{\beta j} \tag{72}$$

Direct calculation shows that

$$'E^i_j = L^i_r E^r_s \overset{-1}{L}{}^s_j \qquad ('\mathbf{E} = \mathbf{L}\mathbf{E}\mathbf{L}^{-1}) \tag{73}$$

under P_{10} gauge transformations. We thus have the following new list of P_{10} invariants:

$$\beta_1 = \mathrm{tr}(\mathbf{E}), \qquad \beta_2 = \mathrm{tr}(\mathbf{E}^2), \qquad \beta_3 = \mathrm{tr}(\mathbf{E}^3), \qquad \beta_4 = \mathrm{tr}(\mathbf{E}^4) \tag{74}$$

In addition to these, there are the mixed invariants

$$\beta_5 = \mathrm{tr}(\mathbf{AE}), \qquad \beta_6 = \mathrm{tr}(\mathbf{A}^2\mathbf{E}), \qquad \beta_7 = \mathrm{tr}(\mathbf{AE}^2) \ldots \tag{75}$$

If we do the same thing starting with the Σ's, we are led to consider the quantities

$$N^i_{uv} = b_u \lrcorner\, b_v \lrcorner\, \Sigma^i = b^j_u b^k_v \Sigma^i_{jk} \tag{76}$$

Under local action of P_{10}, (28) and (37) show that

$$'N^i_{uv} = \overset{-1}{L}{}^p_u \overset{-1}{L}{}^q_v L^i_j N^j_{pq} \tag{77}$$

Accordingly, the N's are invariant under coordinate transformations generated by the local action of $T(4)$, while transforming under the local action of $L(4, R)$ according to (77). The transformation law (77) immediately suggests that we look at

$$F_u = N^v_{uv}, \qquad 'F_u = \overset{-1}{L}{}^j_u F_j \tag{78}$$

It is then an easy matter to construct the P_{10} invariant scalars

$$\begin{aligned} &\rho_1 = F_i h^{ij} F_j, \qquad \rho_2 = F_i A^i_j h^{jk} F_k \\ &\rho_3 = F_i A^i_j A^j_k h^{km} F_m, \qquad \rho_4 = F_i E^i_j h^{jk} F_k \\ &\rho_5 = F_i A^i_j E^j_k h^{km} F_m, \ldots \end{aligned} \tag{79}$$

There are, however, further invariants that can be constructed out of the Σ's. To this end, we set

$$R^i_j = N^i_{uv} N^k_{pq} h^{up} h^{vq} h_{kj} = g^{up} g^{vq} \Sigma^i_{uv} \Sigma^k_{pq} h_{kj} \tag{80}$$

in which case we have

$$'R^i_j = L^i_u R^u_v \overset{-1}{L}{}^v_j \tag{81}$$

We thus have the system of P_{10} invariants

$$\begin{gathered}\eta_1 = \mathrm{tr}(\mathbf{R}), \qquad \eta_2 = \mathrm{tr}(\mathbf{R}^2), \ldots, \qquad \eta_4 = \mathrm{tr}(\mathbf{R}^4) \\ \eta_5 = \mathrm{tr}(\mathbf{RA}), \qquad \eta_6 = \mathrm{tr}(\mathbf{RE}), \ldots \\ \zeta_1 = F_i R^i_j h^{jk} F_k, \qquad \zeta_2 = F_i R^i_j R^j_k h^{km} F_m \\ \zeta_3 = F_i R^i_j A^j_k h^{km} F_m, \qquad \zeta_4 = F_i R^i_j E^j_k h^{km} F_m, \ldots\end{gathered} \tag{82}$$

If we exclude invariants that involve permutation symbols, as required by parity considerations, standard results from invariance theory show that any P_{10} invariant scalar-valued function Π of the B's, θ's, and Σ's is expressible as

$$\Pi = \Pi(\alpha_1, \ldots, \beta_1, \ldots, \rho_1, \ldots, \eta_1, \ldots, \zeta_1, \ldots) \tag{83}$$

It is then a straightforward but laborious job to check that any free field Lagrangian density ΠB satisfies the invariance conditions (62) and (63).

Free field Lagrangian densities of primary interest are those that are at most quadratic in the field tensors (quadratic in the θ's and Σ's). Examining the list of invariants given above, we see that such Lagrangian densities are of the form

$$\bar{V} = B(k_1\alpha_1 + k_2(\alpha_1)^2 + k_3\alpha_2 + k_4\beta_1 + k_5\rho_1 + k_6\eta_1) \tag{84}$$

Here, the k's are coupling constants that are chosen so that the physical dimensions of each term in (84) are those of action per unit spatial volume per unit time. Of the various terms that appear in (84), α_1 has the evaluation

$$\alpha_1 = b^{[u}_m b^{v]}_k \theta^\alpha_{uv} l^k_{\alpha i} h^{mi}$$

which is similar to the linear invariant used by Kibble (1961) and Sciama (1962). Noting that $L(4, R)$ is semisimple, we have

$$l^j_{\alpha i} l^i_{\beta j} = C_{\alpha\beta}$$

where $C_{\alpha\beta}$ are the components of the Cartan–Killing metric on $L(4, R)$. Thus

$$\beta_1 = g^{ab} g^{ij} \theta^\alpha_{ai} C_{\alpha\beta} \theta^\beta_{bj}$$

and

$$\eta_1 = g^{ab} g^{uv} \Sigma^i_{au} \Sigma^j_{bv} h_{ij}$$

which are similar to quadratic free field Lagrangians used in standard Yang–Mills theory. The remaining terms in (89) do not appear to have been used in previous studies.

6. GAUGE CONDITIONS, COORDINATE CONDITIONS, AND THE ANTIEXACT GAUGE

The field equations, constitutive relations, and the frames and coframes of the P_{10} gauge theory are gauge covariant. Accordingly, we are free to apply local Poincaré transformations and their accompanying gauge transformations in order to achieve certain simplifications. In particular, we can effect a gauge transformation such that (35) and (36) may be used to restrict the Yang–Mills compensating 1-forms in useful and often simplifying ways. Restrictions obtained in this way are known as gauge conditions in direct analogy with classic electromagnetic theory. Although the more familiar gauge conditions (Lorentz, Coulomb) are differential, there are strictly algebraic gauge conditions that often prove to be useful. A previous publication (Edelen, 1981) has shown that the antiexact gauge conditions, which are algebraic, lead to marked simplifications in solving the field equations and to intrinsic geometric simplifications that prove to be particularly useful in understanding certain subtilities of the Poincaré gauge theory.

Direct Poincaré gauge theory starts with the Minkowski space M_4, which is globally star shaped with respect to any point in M_4 as center (see Edelen, 1985b). We may therefore apply a global translation so that the center may be identified with the origin of the standard Cartesian coordinate cover of M_4. This construction provides us with a well defined radius vector field $X = x^i\partial_i$. If α is any exterior form on M_4, the linear homotopy operator, H, is defined by (see Edelen, 1985b)

$$H\alpha(x^m) = \int_0^1 X \lrcorner\, \alpha(\lambda x^m)\lambda^{-1}\, d\lambda \tag{85}$$

The notation used in (85) is best explained for the case of a 2-form where

$$\alpha(\lambda x^m) = \alpha_{ij}(\lambda x^m)\lambda\, dx^i \wedge \lambda\, dx^j$$

The linear homotopy operator verifies the identity

$$\alpha = dH\alpha + Hd\alpha \tag{86}$$

for any exterior forms on a star-shaped region (on M_4), and hence any exterior form α has an exact part

$$\alpha_e = dH\alpha$$

and an antiexact part

$$\alpha_a = Hd\alpha$$

Since H is a linear operator, it has a well-defined kernel,

$$A = \{\alpha \in \Lambda(M_4) | H\alpha = 0\}$$

and any $\beta \in A$ is said to be antiexact. Noting that $H^2\alpha = 0$, the antiexact part of any exterior form on M_4 is antiexact and any antiexact form is the antiexact part of some form. Further, A forms a submodule of $\Lambda(M_4)$ and H inverts d on this submodule ($Hd\alpha = \alpha$ for any $\alpha \in A$). The particularly nice thing here is that A can also be characterized by

$$A = \{\alpha \in \Lambda(M_4) | X \lrcorner \alpha = 0, \alpha(0) = 0\} \tag{87}$$

so that antiexact forms are identified through the algebraic conditions

$$X \lrcorner \alpha = 0 \tag{88}$$

and the reference value conditions

$$\alpha(0) = 0 \tag{89}$$

Let $\mathbf{\Gamma}$ be a matrix of connection 1-forms for the local action of $L(4, R)$. We have shown in a previous paper (Edelen, 1981) that we can always find a local Lorentz matrix $\mathbf{\Gamma}$ for which the gauge transformation (35) makes $'\mathbf{\Gamma}$ antiexact. Further, (36) can be used to make $'\boldsymbol{\omega}$ antiexact by choosing

$$\mathbf{t} = H(\mathbf{L}\boldsymbol{\omega} - {}'\mathbf{\Gamma}\mathbf{t}) + \mathbf{k} = H(\mathbf{L}\boldsymbol{\omega}) + \mathbf{k}$$

where $\mathbf{k}$ is a constant column matrix. The last equality obtains because $'\mathbf{\Gamma}$ is antiexact, and hence the module property of antiexact forms shows that $'\mathbf{\Gamma}\mathbf{t}$ is antiexact (belongs to ker H). With this gauge transformation, we have $'\mathbf{x} = \mathbf{L}\mathbf{x} + \mathbf{t}$, and hence the image of the origin is given by $'\mathbf{x}(0) = \mathbf{t}(0)$. Accordingly, we can achieve $'\mathbf{x}(0) = \mathbf{0}$ by an appropriate choice of $\mathbf{k}$ [i.e., $\mathbf{k} = -H(\mathbf{L}\boldsymbol{\omega})(0) = 0$]. The new and old coordinate covers may thus be chosen so that they agree at the origin, a fact that will assume particular importance in just a minute. These considerations also show that implementation of the antiexact gauge conditions that obtain by the above procedure may also be viewed as imposing an explicit system of coordinate conditions (i.e., an explicit choice of $'\mathbf{x}$ in terms of $\mathbf{x}$). The latter are essential in gravitation theory, as is well known.

Having noted these facts, we assume that the gauge transformation generated by $\mathbf{L}$ and $\mathbf{t}$ has been effected so that we may drop the primes. The resulting compensating 1-forms W^α and ϕ^i will now satisfy the antiexact gauge conditions

$$X \lrcorner W^\alpha = 0, \qquad W^\alpha(0) = 0 \tag{90}$$

$$X \lrcorner \phi^i = 0, \qquad \phi^i(0) = 0 \tag{91}$$

Written out in component form, these conditions become

$$x^j W^\alpha_j(x^m) = 0, \qquad x^j \phi^i_j(x^m) = 0 \tag{92}$$

$$W^\alpha_i(0) = 0, \qquad \phi^i_j(0) = 0 \tag{93}$$

The first set, (92), is a system of explicit algebraic conditions that must be satisfied by the fields at each point. The second set, (93), may be viewed as selecting reference values of zero for the fields at the origin. It is useful to note in this context that the Yang and Wu (1969) solution of the free SU(2) gauge field equations satisfies (92).

The antiexact gauge conditions carry particularly useful information. We first note that (9) and (13) show that an evaluation at the origin gives

$$B^i_j(0) = \delta^i_j, \qquad b^i_j(0) = \delta^i_j \tag{94}$$

and hence

$$B^i(0) = dx^i, \qquad b_i(0) = \partial_i \tag{95}$$

The frame fields b_i and the coframe fields B^i induced by minimal replacement thus coincide with the natural frame and coframe fields of M_4 at the origin. In addition, (11) and (15) show that

$$g_{ij}(0) = h_{ij}, \qquad g^{ij}(0) = h^{ij} \tag{96}$$

and hence the metric structure induced by minimal replacement and the metric structure of M_4 also agree at the origin.

Let U_4 denote the four-dimensional manifold whose geometric structure is that induced by minimal replacement for P_{10}. The space U_4 thus has the preferred coordinate cover $'\mathbf{x}$ that is obtained from the coordinate cover of M_4 by local action of the element of P_{10} that achieves the antiexact gauge conditions. Since the two coordinate covers agree at the origins, U_4 and M_4 may be viewed as attached to each other at their origins. Further, (95) show that the tangent spaces and the cotangent spaces of U_4 and M_4 may also be viewed as attached at the fibers over the origins. Finally, (96) show that the metric structures of U_4 and M_4 are coincident over the origins. Accordingly, we may view U_4 and its geometric structure as attached to M_4 and its geometric structure in a natural way.

Everything in the direct gauge theory of P_{10} arises out of M_4 and its geometry and fields by minimal replacement, the space U_4 included. There is thus the fundamental question of whether physical space time remains M_4, or becomes U_4. On the surface, either of these two interpretations is tenable, even though we are accustomed to think of gravity as arising from the curvature properties of physical space-time. However, the metric tensor defined by (11) can be viewed as a field on M_4 from the standpoint of Poincaré gauge theory, and likewise the curvature associated with this metric tensor becomes a system of fields defined over M_4. This interpretation is not available in general relativity because the physics did not start out in Minkowski space and become modified by minimal replacement for P_{10} on Minkowski space.

7. VANISHING TORSION AS WEAK CONSTRAINTS

One of the essential differences between Einstein's gravitation theory and the gauge theory of P_{10} is that minimal replacement of the gauge theory leads to a manifold U_4 with both curvature and torsion, while the Einstein theory has only curvature. It is thus clear that torsion free solutions of the P_{10} gauge field theory assume particular importance.

A combination of (7) and (17) gives the following explicit evaluation of the torsion 2-forms:

$$\Sigma^i = \theta^\alpha l^i_{\alpha j} x^j + d\phi^i + W^\alpha l^i_{\alpha j} \wedge \phi^j \tag{97}$$

We are thus interested in those situations in which

$$d\phi^i = -\theta^\alpha l^i_{\alpha j} x^j - W^\alpha l^i_{\alpha j} \wedge \phi^j \tag{98}$$

that is,

$$\Sigma^i = 0 \tag{99}$$

There are two ways in which (99) can be realized. In the first instance, (99) are viewed as strong constraints, in which case they must be included in the basic variational principle by means of Lagrange multipliers. If this were done, (99) would be enforced for all solutions of the field equations and the theory would be identically torsion free, but involve Lagrange multipliers for the strong constraints (99). A step as drastic as this seems unwarranted at the present time. The second way is to view (99) as restrictions that are used to single out solutions of the field equations with the property of vanishing torsion; that is, (99) are weak constraints that are to be applied after the field equations are derived from the basic variational principle. It is this latter alternative that we pursue in this section.

In order that (99) hold, the translation compensating fields ϕ^i must satisfy the exterior differential equations (98). Now, (98) entail the integrability conditions

$$\theta^\alpha l^i_{\alpha j} \wedge B^j = 0 \tag{100}$$

of which only 16 are independent (4-forms of degree 3 on a four-dimensional space). Progress from this point on is particularly facilitated by use of the antiexact gauge. The ϕ's and the W's are thus antiexact 1-forms that belong to the kernel of the linear homotopy operator H introduced in the previous section. Since ϕ^i is antiexact, $\phi^i = Hd\phi^i$, and hence (98) gives

$$\phi^i = -H(\theta^\alpha l^i_{\alpha j} x^i) \tag{101}$$

because

$$W^\alpha l^i_{\alpha j} \wedge \phi^j$$

is antiexact by the module property of antiexact forms and hence belongs to ker H. Similarly, when (6) is substituted into (101), we obtain

$$\phi^i = -H(dW^\alpha l^i_{\alpha j} x^j) \tag{102}$$

Finally, we note that

$$dW^\alpha l^i_{\alpha j} x^j = d(W^\alpha l^i_{\alpha j} x^j) + W^\alpha l^i_{\alpha j} \wedge dx^j$$

and that

$$Hd(W^\alpha l^i_{\alpha j} x^j) = W^\alpha l^i_{\alpha j} x^j$$

because

$$W^\alpha l^i_{\alpha j} x^j$$

is antiexact. When these results are put into (102), we obtain the explicit evaluation

$$\phi^i = -W^\alpha l^i_{\alpha j} x^j - H(W^\alpha l^i_{\alpha j} \wedge dx^j) \tag{103}$$

The translation compensating fields ϕ^i are thus uniquely determined in terms of the $L(4, R)$ compensating fields W^α whenever we have solutions that are torsion free.

The first thing to note is that (9) gives

$$B^i = dx^i - H(W^\alpha l^i_{\alpha j} \wedge dx^j) \tag{104}$$

Thus, the distortion 1-forms differ from the natural basis elements for 1-forms on M_4 by antiexact 1-forms that are uniquely determined by the W's that compensate for local action of $L(4, R)$. It thus follows from (11) that the metric tensor on U_4 is also uniquely determined by the W's. The metric differential structure of U_4, including its Riemann curvature tensor, is therefore uniquely determined by the compensating 1-forms for the Lorentz sector [for $L(4, R)$].

There is an alternative point of view that is also useful here. Combining (16) and (99), the conditions for vanishing torsion read

$$DB^i = dB^i + W^\alpha l^i_{\alpha j} \wedge B^j = 0 \tag{105}$$

The ideal of $\Lambda(M_4)$ that is generated by the distortion 1-forms is thus a closed ideal and hence the Frobenius theorem shows that there exists a nonsingular matrix $\mathbf{A}$ of functions and a system of independent functions $\{p^i(x^k))|1 \le i \le 4\}$ such that

$$B^i = A^i_j \, dp^j \tag{106}$$

When this representation is substituted into (11), the metric tensor is evaluated by

$$g_{ij} = A^u_a h_{uv} A^v_b \, \partial_i p^a \, \partial_j p^b \tag{107}$$

Thus, introduction of new coordinates by $x^{i'} = p^i(x^k)$ leads to the metric tensor

$$g'_{ab} = A^u_a h_{uv} A^v_b \tag{108}$$

with the obvious simplifications. It is of interest to note in this context that conditions similar to (105) occur in most treatments of Poincaré gauge theory reported in the literature.

We now turn to the field equations. It was shown in Section 6 that there are no free field Lagrangian densities that are linear in the components of Σ^k. Thus (20) and $\Sigma^k = 0$ show that

$$G^{ij}_k = 0 \tag{109}$$

and that we may put all components of the torsion equal to zero in $V = B\Pi$ before evaluating the remaining constitutive relations. Now, the ϕ's enter only through the distortion 1-forms while Π depends on the B's only through the b's. Thus (13), the second of (20), and algebraic simplification give us

$$S^i_k = Bb^i_u(\Pi \delta^u_k - b^r_k \, \partial\Pi/\partial b^r_u) \tag{110}$$

When these results are substituted into the field equations (23), (24) and use is made of (22), the field equations in the absence of torsion become

$$T^i_k = \Pi \delta^i_k - b^r_k \frac{\partial \Pi}{\partial b^r_i} \tag{111}$$

$$dH_\alpha - W^\gamma C^\beta_{\gamma\alpha} \wedge H_\beta = Bb^j_k L^k_A M^A_{\alpha E} \Psi^E \mu_j \tag{112}$$

where we have set

$$H_\alpha = H^{ij}_\alpha \mu_{ij} = B \frac{\partial \Pi}{\partial \theta^\alpha_{ij}} \mu_{ij} \tag{113}$$

Let us first note that the torsion-free condition gives

$$\Pi = \Pi(b^i_j, \theta^\alpha_{ij}) \tag{114}$$

and hence the right-hand sides of (111) depend on the b's and the $L(4, R)$ curvatures θ^α, but not on derivatives of the $L(4, R)$ curvature expressions. Thus, since the T's are the components of the gauge momentum energy complex, the field equations (111) are of exactly the same form as Einstein's field equations: {momentum energy} = {curvature}. There is thus the strong expectation that Einstein's equations for the gravitational field can be obtained by appropriate choice of Π.

This conclusion may be a little unsettling on first reading, for we are accustomed to think of gravity as described by second-order differential equations in terms of the metric tensor as "potential," while the curvature quantities θ^α contain only first derivatives of the field quantities W^α. However, (11) shows that the metric is quadratic in the W's while we also have the additional field equations (112) to solve. In fact, we may look upon (111) and (112) as a system of first-order differential equations whose combination and elimination will result in the second-order differential equations that obtain in the Einstein theory. This situation is similar to what happens in electrodynamics where the field equations are first-order differential equations in **E** and **B** but result in second-order differential equations in terms of the vector potential.

The simplest way of understanding the content of the field equations (112) is to write them in the equivalent form

$$dH_\alpha = W^\beta C^\gamma_{\beta\alpha} \wedge H_\gamma + J_\alpha \tag{115}$$

where we have used J_α for the spin currents of the matter field that occur on the right-hand side of (112). Since $H_\alpha = dH(H_\alpha) + H(dH_\alpha)$, where H is the linear homotopy operator, (115) is equivalent to the Riemann–Graves integral equation

$$H_\alpha = d\sigma_\alpha + H(W^\beta C^\gamma_{\beta\alpha} \wedge H_\gamma + J_\alpha) \tag{116}$$

where σ_α are antiexact 1-form that remain to be determined. With the antiexact gauge, W^β is antiexact and an iteration of (116) together with the fact that antiexact forms form a submodule that is the kernel of H give us

$$H_\alpha = d\sigma_\alpha + H(W^\beta C^\gamma_{\beta\alpha} \wedge d\sigma_\gamma + J_\alpha) \tag{117}$$

Thus, if the matter fields have no spin current, as in the original Einstein theory, we have

$$H_\alpha = d\sigma_\alpha + H(W^\beta C^\gamma_{\beta\alpha} \wedge d\sigma_\gamma) \tag{118}$$

Now, H_α are determined in terms of W^β and dW^β by (113), and hence (111) and (118) may be viewed as a system of integrodifferential equations for the determination of W^β and σ_α. These are the field equations for gravitational phenomenae in the P_{10} gauge theory in those cases in which there is direct correspondence with the assumptions of the Einstein theory. Finally, we note that the integrability conditions (100) would seem to demand that $J_\alpha = 0$, because matter fields could be "turned on" that would lead to determinations of the $L(4, R)$ curvatures through (111) and (117) that would violate (100).

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